A PROPORTIONAL EXTENSION OF THE SHAPLEY VALUE FOR MONOTONE GAMES WITH A COALITION STRUCTURE

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Abstract

The Owen value is a modification of the Shapley value for games with a coalition structure. In this paper, we propose another modification of the Shapley value for monotone games with a coalition structure. This new value is a double-extension of the Shapley value in the next sense: the amount obtained by an union coincides with the Shapley value of the union in the quotient game, and the players of each union share this amount proportionally to their Shapley values in the game without unions. We give two characterizations of this new value. The axiomatic systems used here can be compared with parallel axiomatizations of the Owen value.
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1 Introduction

The assessment of the strategic position of each player in any game is a main objective of the cooperative game theory, as it can be applied to e.g. sharing costs or profits in economic problems or measuring the power of each agent in a collective decision–making system. The *Shapley value* φ is the best known concept in this respect, and its axiomatic presentation (Shapley [10], also in Roth [9]) introduced a new, elegant style in game theory and opened a fruitful research line.

Forming coalitions is a most natural behavior in cooperative games, and the evaluation of the consequences that derive from this action is also of great interest to game theorists. *Games with a coalition structure* were first considered by Aumann and Drèze [4], who extended the Shapley value to this new framework in such a

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manner that the game really splits into subgames played by the unions isolatedly from each other, and every player receives the payoff allocated to him by the Shapley value in the subgame he is playing within his union. A second approach was used by Owen [7] (also in Owen [8]), when introducing and axiomatically characterizing his coalitional value Φ (*Owen value*). In this case, the unions play a *quotient game* among themselves, and each one receives a payoff which, in turn, is shared among its players in an internal game. Both payoffs, in the quotient game for unions and within each union for its players, are given by the Shapley value. In addition to the initial one, many other axiomatic characterizations of Φ can be found in the literature (Hart and Kurz [6], Winter [12], Amer and Carreras [2] and [3], Vázquez–Brage et al. [11], Hamiache [5] or Albizuri [1] among others).

In this paper, we consider another extension of the Shapley value for monotone games with a coalition structure. As Owen did for the Owen value, the Shapley value is used twice. The first one is in the quotient game, in which each union receives the Shapley value in this game, that is, this first step is the same for the Owen value and the new value proposed in this paper. In the second step, we consider that all players of an union should profit equally for joining an union, that is, the payoffs of players of the same union in the games with or without unions must be proportional. These proportions are given by the Shapley value in the game without unions.

Besides, we provide several axiomatic characterizations for this new coalitional value that are able to be compared with some of the existing ones for the Owen value. In the characterizations of the new solution, two properties of proportionality within unions play an important role.

The organization of the paper is then as follows. In Section 2, a minimum of preliminaries is provided. In Section 3 we motivate and define the new value. Section 4 is devoted to characterizing it.

2 Preliminaries

Although the reader is assumed to be generally familiar with the cooperative game theory, we recall here some basic notions.

2.1 Games and values

A finite transferable utility cooperative game (from now on, simply a *game*) is a pair (N, v) defined by a finite set of *players* N, usually $N = \{1, 2, \ldots, n\}$, and a function $v : 2^N \rightarrow \mathbb{R}$, that assigns to each *coalition* $S \subseteq N$ a real number $v(S)$ and satisfies $v(\emptyset) = 0$. In the sequel, $\mathcal{G}_N$ will denote the family of all games on a given N and $\mathcal{G}$ will denote the family of all games. A game (N, v) is *monotone* if $v(S) \leq v(T)$, for all $S \subseteq T \subseteq N$. $\mathcal{G}_N^+$ will denote the family of all monotone games on a given N and $\mathcal{G}^+$ will denote the family of all monotone games.

A *simple game* is a monotone game (N, v) such that $v(S) = 1$ or $v(S) = 0$ for every $S \subseteq N$, and $v(N) = 1$. Given $S \subseteq N$, the *unanimity game* (N, u_S) is the simple game such that $u_S(T) = 1$ if and only if $S \subseteq T$. A simple game (N, v) is a *weighted* *majority game* if there exist a set of *weights* $w_1, w_2, ..., w_n$ for players, with $w_i \geq 0$ for $1 \leq i \leq n$, and a *quota* $q \in \mathbb{R}^+$ such that $v(S) = 1$ if and only if $w(S) \geq q$, where $w(S) = \sum_{i \in S} w_i$. A representation of a weighted majority game is given by $[q; w_1, w_2, ..., w_n]$.

A player $i \in N$ is a *dummy* in game (N, v) if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, that is, if all his marginal contributions equal $v(\{i\})$. A player $i \in N$ is a *null player* in game (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. Two players $i, j \in N$ are *symmetric* in game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$, i.e., if their marginal contributions to each coalition coincide.

By a *value* we will mean a map f that assigns to every game $(N, v) \in \mathcal{G}_N$ a vector $f(N, v) \in \mathbb{R}^N$ with components $f_i(N, v)$ for all $i \in N$. If the value is defined on $\mathcal{G}^+$, we will require that $f_i(N, v) \geq 0$ for all $i \in N$.

Definition 2.1 (Shapley [10]) The *Shapley value* φ is the value defined by

$$\varphi_i(N, v) = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{(n-s)\binom{n}{s}} \Big[v(S \cup \{i\}) - v(S) \Big] \text{ for any } i \in N \text{ and any } (N, v) \in \mathcal{G}, \tag{1}$$

where $s = |S|$.

2.2 Games with a coalition structure

Let us consider a finite set, say, $N = \{1, 2, \ldots, n\}$. We will denote by $P(N)$ the set of all partitions of N. Each $P \in P(N)$, of the form $P = \{P_1, P_2, \ldots, P_m\}$, is called a *coalition structure* or *system of unions* on N. The so–called *trivial coalition structures* are $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$, where each union is a singleton, and $P^N = \{N\}$, where the *grand coalition* forms. If $P_k \in P$ and $j \in P_k$, P_{-j} will denote the partition obtained from P when player j leaves union P_k and becomes isolated, i.e.,

$$P_{-j} = \{P_h \in P : h \neq k\} \cup \{P_k \backslash \{j\}, \{j\}\}.$$

A *cooperative game with a coalition structure* is a triple (N, v, P) where $(N, v) \in \mathcal{G}$ and $P \in P(N)$. The set of all cooperative games with a coalition structure will be denoted by $\mathcal{G}^{cs}$, and by $\mathcal{G}_N^{cs}$ the subset where N is the player set. The set of all monotone cooperative games with a coalition structure will be denoted by $\mathcal{G}^{+cs}$, and by $\mathcal{G}_N^{+cs}$ the subset where N is the player set.

If $(N, v, P) \in \mathcal{G}^{cs}$ and $P = \{P_1, P_2, \ldots, P_m\}$, the *quotient game* (M, v^P) is the cooperative game played by the unions, or, rather, by the set $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^P(R) = v(\bigcup_{r \in R} P_r) \quad \text{for all } R \subseteq M. \tag{2}$$

Notice that (M, v^P) is nothing but (N, v) whenever $P = P^n$. If a game $(N, v, P) \in \mathcal{G}^{+cs}$ the quotient game $(M, v^P) \in \mathcal{G}^+$.

Given $(N,v,P) \in \mathcal{G}^{cs}$ and $P = \{P_1, P_2, \ldots, P_m\}$, a union $P_k \in P$ is a *null union* if k is a null player in (M, v^P) and two unions $P_k, P_s \in P$ are *symmetric unions* if k and s are symmetric players in (M, v^P). By a *coalitional value* we will mean a map g that assigns to every game with a coalition structure (N,v,P) a vector $g(N,v,P) \in \mathbb{R}^N$ with components $g_i(N,v,P)$ for each $i \in N$. If the coalitional value is defined on $\mathcal{G}^{+cs}$, we will require that $g_i(N,v,P) \geq 0$ for all $i \in N$.

Definition 2.2 (Owen [7]) The *Owen value* Φ is the coalitional value defined by

$$\Phi_i(N,v,P) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{(m-r)\binom{m}{r}} \frac{1}{(p_k - t)\binom{p_k}{t}} \Big[v(Q \cup T \cup \{i\}) - v(Q \cup T) \Big] \tag{3}$$

for all $i \in N$ and all $(N,v,P) \in \mathcal{G}^{cs}$, where $P_k \in P$ is the union such that $i \in P_k$, $m = |M|$, $p_k = |P_k|$, $r = |R|$, $t = |T|$, and $Q = \bigcup\limits_{r \in R} P_r$.

Definition 2.3 Given a value f on $\mathcal{G}$, a coalitional value g on $\mathcal{G}^{cs}$ is a *coalitional f–value* if

$$g(N,v,P^n) = f(N,v) \quad \text{for all} \quad (N,v) \in \mathcal{G}. \tag{4}$$

The Owen value is a coalitional Shapley value, that is, $\Phi(N,v,P^n) = \varphi(N,v)$. Besides, the Owen value satisfies the following properties, that we state for a coalitional value g.

A1. (*Efficiency*) For all $(N,v,P) \in \mathcal{G}^{cs}$, $\sum_{i \in N} g_i(N,v,P) = v(N)$.

A2. (*Dummy player*) If $i \in N$ is a dummy in (N,v) then $g_i(N,v,P) = v(\{i\})$.

A3. (*Null player*) If $i \in N$ is a null player in (N,v) then $g_i(N,v,P) = 0$.

A4. (*Symmetry in the unions*) If $i, j \in P_k \in P$ are symmetric players in (N,v) then $g_i(N,v,P) = g_j(N,v,P)$.

A5. (*Equal marginal contributions*) If (N,v) and (N,w) are games with a common player set N, and some player $i \in N$ satisfies $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, then $g_i(N,v,P) = g_i(N,w,P)$.

A6. (*Symmetry in the quotient*) If $k, s \in M$ are symmetric players in (M, v^P) then $\sum_{i \in P_k} g_i(N,v,P) = \sum_{j \in P_x} g_j(N,v,P)$.

A7. (*Balanced contributions for the unions*) If $(N,v,P) \in \mathcal{G}^{cs}$ and $i, j \in P_k \in P$ are distinct players, then

$$g_i(N,v,P) - g_i(N,v,P_{-j}) = g_j(N,v,P) - g_j(N,v,P_{-i}).$$

A8. (*Quotient game property*) If $(N,v,P) \in \mathcal{G}^{cs}$ and $P_k \in P$, then

$$\sum_{i \in P_k} g_i(N,v,P) = g_k(M, v^P, P^m).$$

A9. *(Additivity).* If $(N, v, P), (N, w, P) \in \mathcal{G}^{cs}$, $P_k \in P$, and $i \in P_k$, then

$$g_i\left(N, v + w, P\right) = g_i\left(N, v, P\right) + g_i\left(N, w, P\right).$$

We add a comment about balanced contributions for the unions property, which can be found on [11].

This property states that if two players i and j are in the same a priori union, then the loss (or gain) that player i inflicts on player j when he leaves the union is the same as the loss (or gain) inflicted on player i when j leaves the union. This property reflects the idea that all players in a union should profit equally for joining the union and hence that it cannot be the case that one specific player extracts all the benefits that are generated by the formation of the union.

In the sequel, A0 will mean the conjunction of A1, A2, A4 and A5 if we consider these properties as defined only for the trivial system of unions P^n. This will make our statements simpler. We will first establish a close relationship between the coalitional values satisfying A0 and the Shapley value. More precisely:

Proposition 2.4 *A coalitional value g satisfies A0 if, and only if, it is a coalitional Shapley value, i.e.*

$$g_i(N, v, P^n) = \varphi_i(N, v) \quad \textit{for all} \quad i \in N \quad \textit{and all} \quad (N, v) \in \mathcal{G}. \tag{5}$$

Proof. The proof follows the same guidelines as Young's [13] proof. The only difference between Young's statement and ours is that he is talking about values, whereas we are referring to coalitional values: the connection is given by the appearance of the trivial coalition structure P^n in our axiom set A0. □

In the next two theorems, we recall two characterizations of the Owen value by means of some of the previous properties.

Theorem 2.5 *(Owen [7]) A coalitional value satisfies A1, A3, A4, A6, and A9 if, and only if, it is the Owen value Φ.*

Theorem 2.6 *(Vázquez–Brage et al. [11]) A coalitional value satisfies A0, A7, and A8 if, and only if, it is the Owen value Φ.*

3 A new value for monotone games with a coalition structure

In this section, we define a new value for monotone games with a coalition structure. We will call to this solution the proportional coalitional Shapley value. We feel that one should strongly prevent from being dogmatic when considering values, as solution concepts for cooperative games. Probably, there is no value able to cover all situations. For example, there is no unanimous criterion to choose among using either the Shapley value φ or the Banzhaf value β as power index *in all cases*. Then, we contend that pure and applied game theorists should be flexible at most in this respect. In both theory and practice, one has often to handle additional information not stored in either the characteristic function v of the game or the coalition structure when evaluating this couple. Even those values that appear as the best placed in this sense might well be conditioned by the characteristics of the problem where we pretend to use them. The history of science is full of examples of theoretical models that only after a certain period of time have been proven to be useful in practice.

Let $(N, v, P) \in \mathcal{G}^{+cs}$ with $P = \{P_1, P_2, \ldots, P_m\}$. How can we assess the payoff of a player $i \in P_k$ in this game by means of a coalitional Shapley value? First, the unions play the quotient game (M, v^P) and the union P_k obtains a payoff equal to $\varphi_k \left(M, v^P\right)$. We agree with the statement of the previous section about the balanced contributions for the unions property, that is, players in an union should profit equally for joining the union. Taking this fact into account, the amount $\varphi_k \left(M, v^P\right)$, assigned to the union P_k must be shared proportionally among the players of the union. For the proportional coalitional Shapley value, the proportionality is given by the payoffs in the game without unions. If these amounts are given by the Shapley value, player i obtains the proportion $\varphi_i (N, v) / \sum_{j \in P_k} \varphi_j (N, v)$ of the payoff of the union, $\varphi_k \left(M, v^P\right)$.

We formalize this idea in the next definition.

Definition 3.1 The *proportional coalitional Shapley value AC* is the coalitional value on $\mathcal{G}^{+cs}$ defined by

$$AC_i(N, v, P) = \begin{cases} \varphi_k \left(M, v^P\right) \frac{\varphi_i(N,v)}{\sum_{j \in P_k} \varphi_j(N,v)} & \text{if} \quad P_k \quad \text{is not a null union} \\ 0 & \text{if} \quad P_k \quad \text{is a null union} \end{cases} \tag{6}$$

for all $i \in N$ and all $(N, v, P) \in \mathcal{G}^{+cs}$, where $P_k \in P$ is the union such that $i \in P_k$.

Following the previous procedure, the proportional coalitional Shapley value gives to a coalition its Shapley value in the quotient game, and this payoff is splitted among the players of this union proportionally. The weights for this proportion are given by the Shapley value of the game without unions.

The main principle of the new value is directly related to the concept of balanced contributions for the unions property, that is, players of an union should profit equally for joining this union. We agree with this idea, but we take this equality defined as a proportion and not as a difference. The gain (or loss) that for a player supposes forming a union is given by the quotient between his payoff in the game with unions and the games without unions. As we want this gain (or loss) to be the same for all players in each union, we equal these quotients.

4 Two axiomatic approaches

For any value, understood as a solution concept for cooperative conflicts, it is always interesting to have an explicit formula and also a list of properties of the value, as long as possible.

Besides, it is desirable to provide axiomatic systems of the value and, in many cases, parallel axiomatic characterizations with other values. There are some reasons for this interest of game theorists in getting them. First, for a mathematically elegant and pleasant spirit. Second, because a set of basic (and assumed independent and hence minimal) properties is a most convenient and economic tool to decide on the use of the value. Finally, *parallel* axiomatic characterizations are especially interesting because they favor the easiness when comparing different options to be chosen as the preferred value, that is, they allow the researcher to compare a given value with others and select the most suitable one for the problem he or she is facing each time.

A final comment: only a few properties found in the literature can really be considered absolutely compelling, i.e. almost no axiom is compelling *in vacuo,* but only inserted in the framework of a given, specific cooperative conflict. The conclusion is that all of us should look at axioms with an open mind and without a priori value judgements.

We shall consider two new properties for a coalitional value g on $\mathcal{G}^{+cs}$.

A10. (*Proportionality within unions*) If $(N, v, P) \in \mathcal{G}^{+cs}$, $P_k \in P$, and $i, j \in P_k$ are distinct players, then

$$g_i(N, v, P)g_j(N, v, P^n) = g_j(N, v, P)g_i(N, v, P^n).$$

A11. *(Weighted additivity).* If $(N, v, P), (N, w, P) \in \mathcal{G}^{+cs}$, $P_k \in P$ is a non null union in $\left(M, v^P\right)$ and $\left(M, w^P\right)$, and $i \in P_k$, then

$$g_i\left(N, v+w, P\right) \times h_k\left(N, v+w\right) =$$

$$g_i\left(N, v, P\right) \times h_k\left(N, v\right) + g_i\left(N, w, P\right) \times h_k\left(N, w\right),$$

where

$$h_k\left(N, v\right) = \frac{\sum_{i \in P_k} g_i\left(N, v, P^n\right)}{\sum_{i \in P_k} g_i\left(N, v+w, P^n\right)}$$

and $h_k\left(N, w\right)$ and $h_k\left(N, v+w\right)$ are defined analogously.

In the first property resides the main concept of the proportional coalitional Shapley value. Two players of the same a priori union obtain proportional payoffs in the games with or without unions.

As to the second, let us consider two monotone games with a priori unions and the relation between the value of a player and the value of his union. The weighted additivity property establishes a relation among the sum game and the original two games, where the weights are given by the proportion between the amounts given by the value to players of an union in the game without unions and the amount given by the value to the union in the game played by the unions. It is important to point out that these weights do not depend on the system of unions.

In the next result, we give a characterization of the proportional coalitional Shapley value, very similar to that proposed by Owen for the Owen value. The only difference lies in the property of additivity used for the Owen value whereas the proportional coalitional Shapley value satisfies weighted additivity.

Theorem 4.1 *(Existence and uniqueness) A coalitional value defined on $\mathcal{G}^{+cs}$ satisfies A1, A3, A4, A6, and A11 if, and only if, it is the proportional coalitional Shapley value AC.* $\square$

Proof. (a) (Existence) We prove that the proportional coalitional Shapley value AC satisfies the properties of the theorem. Given $(N, v, P) \in \mathcal{G}^{+cs}$

A1. Taking into account that

$$\sum_{i \in N} AC_i\,(N, v, P) = \sum_{k \in M} \sum_{i \in P_k} AC_i\,(N, v, P)$$

it holds that

$$\sum_{k \in M} \sum_{i \in P_k} \varphi_k\left(M, v^P\right) \frac{\varphi_i\,(N, v)}{\sum_{j \in P_k} \varphi_j\,(N, v)} = \sum_{k \in M} \varphi_k\left(M, v^P\right) = v^P(M),$$

by the efficiency of the Shapley value. Finally $v^P(M) = v(N)$.

A3. If $i \in P_k$ is a null player in (N, v) then $AC_i\,(N, v, P) = 0$ if P_k is a null union. If P_k is not a null union, it holds that $\varphi_i\,(N, v) = 0$, and then $AC_i\,(N, v, P) = 0$.

A4. If $i, j \in P_k \in P$ are symmetric players, as the Shapley value is symmetric it holds that $AC_i\,(N, v, P) = AC_j\,(N, v, P)$.

A6. If $P_k, P_s \in P$ are two (non null) symmetric unions, as

$$\sum_{i \in P_k} AC_i\,(N, v, P) = \varphi_k\left(M, v^P\right) \text{ and } \sum_{i \in P_s} AC_i\,(N, v, P) = \varphi_s\left(M, v^P\right)$$

and the Shapley value is symmetric, $\sum_{i \in P_k} AC_i\,(N, v, P) = \sum_{i \in P_s} AC_i\,(N, v, P)$. If the unions are null the result is trivial.

A11. Consider another game $(N, w, P) \in \mathcal{G}^{+cs}$. Then, for all $i \in P_k$ such that P_k is a non null union in $\left(M, v^P\right)$ and $\left(M, w^P\right)$,

$$AC_i\,(N, v + w, P) \times h_k\,(N, v + w) =$$

$$\varphi_k\left(M, v^P + w^P\right) \times \frac{\varphi_i\,(N, v + w)}{\sum_{j \in P_k} \varphi_j\,(N, v + w)} \times \frac{\sum_{i \in P_k} AC_i\,(N, v + w, P^n)}{AC_k\,(M, v^P + w^P, P^m)} =$$

$$\varphi_k\left(M, v^P + w^P\right) \times \frac{\varphi_i\,(N, v + w)}{\sum_{j \in P_k} \varphi_j\,(N, v + w)} \times \frac{\sum_{i \in P_k} \varphi_i\,(N, v + w)}{\varphi_k\,(M, v^P + w^P)} =$$

$$\varphi_i\,(N, v + w) = \varphi_i\,(N, v) + \varphi_i\,(N, w) =$$

$$AC_i\,(N, v, P) \times h_k\,(N, v) + AC_i\,(N, w, P) \times h_k\,(N, w)\,.$$

(b) (Uniqueness) Given $k > 0$, let us consider the game $(N, ku_S, P) \in \mathcal{G}^{+cs}$. If a coalitional value f on $\mathcal{G}^{+cs}$ satisfies A1, A3, A4, and A6, then

$$f_i\,(N, ku_S, P) = \begin{cases} \frac{k}{r_k \times r} & \text{if } i \in P_k \cap S \\ 0 & \text{if } i \notin S, \end{cases}$$

where r_k and r are the cardinalities of $R_k = P_k \cap S$ and $R = \{k \in M / P_k \cap S \neq \varnothing\}$, respectively.

Now we prove that the solution is unique for games with the trivial system of unions $(N, v, P^n) \in \mathcal{G}^{+cs}$. As the solution f satisfies A11, given two games $(N, v, P^n), (N, w, P^n) \in \mathcal{G}^{+cs}$ it holds that

$$f_i(N, v+w, P^n) \times h_k(N, v+w) =$$

$$f_i(N, v+w, P^n) \times \frac{\sum_{i \in P_k} f_i(N, v+w, P^n)}{f_k(M, v^P + w^P, P^m)} = f_i(N, v+w, P^n) =$$

$$f_i(N, v, P^n) \times \frac{\sum_{i \in P_k} f_i(N, v, P^n)}{f_k(M, v^P, P^m)} + f_i(N, w, P^n) \times \frac{\sum_{i \in P_k} f_i(N, w, P^n)}{f_k(M, w^P, P^m)} =$$

$$f_i(N, v, P^n) + f_i(N, w, P^n),$$

i.e., the solution f is additive when we only consider games with trivial system of unions P^n.

If v is a monotone game, it can be written in this way:

$$v = \sum_{\substack{S \subseteq N \\ S \neq \varnothing}} c_S u_{S.}$$

If we consider the games v^+ and v^- defined by

$$v^+ = \sum_{\substack{S \subseteq N \\ c_S \geq o}} c_S u_S \quad \text{and} \quad v^- = \sum_{\substack{S \subseteq N \\ c_S < o}} -c_S u_S$$

it holds that $v + v^- = v^+$. These three games are monotone and, besides, v^+ and v^- can be written as a sum of games of the form ku_S with $k > 0$. As the value f is unique for games $(N, ku_S, P^n) \in \mathcal{G}^{+cs}$, where $k \geq 0$, and f is additive for games with a trivial system of unions (N, v, P^n), the solution f is unique for the games (N, v^+, P^n) and (N, v^-, P^n). Taking into account the equality $v + v^- = v^+$ and the property of weighted additivity (which for these games coincides with the additivity property), the solution f is unique for games (N, v, P^n).

Finally, let $(N, v, P) \in \mathcal{G}^{+cs}$. Taking into account that f satisfies the weighted additivity property, that it is unique for games (N, v, P^n), and that it can be obtained from the equality $v + v^- = v^+$ for any game v, it holds that the solution f must be unique. □

Now, we state and prove our second characterization.

Theorem 4.2 *(Existence and uniqueness) A coalitional value g defined on $\mathcal{G}^{+cs}$ satisfies A0, A8, and A10 if, and only if, it is the proportional coalitional Shapley value AC. In other words, AC is the unique coalitional Shapley value defined on $\mathcal{G}^{+cs}$ that satisfies A8 and A10.*

Proof. (a) (Existence) 1. The proportional coalitional Shapley value AC satisfies A0, that is, A1, A2, A4 and A5 for $P = P^n$. According to Proposition 2.4, it suffices to check equation (5). Let $(N, v) \in \mathcal{G}^+$ and $i \in N$. As we deal with $P = P^n$, we have $M = N$, $P_k = \{i\}$ (so that $k = i$ and $|p_k| = 1$), $T = \emptyset$ and $Q = R$, when applying formula (6). Thus

$$AC_i(N, v, P) = \varphi_i(N, v).$$

2. The proportional coalitional Shapley value AC satisfies A10, the property of proportionality whiting unions. Let $(N, v, P) \in \mathcal{G}^{+cs}$, $P_k \in P$, and $i, j \in P_k$ be distinct players. Then

$$AC_i\left(N, v, P\right) \times AC_j\left(N, v, P^n\right) = AC_i\left(N, v, P^n\right) \times AC_j\left(N, v, P\right).$$

3. The proportional coalitional Shapley value AC satisfies A8, the quotient game property. Let $(N, v, P) \in \mathcal{G}^{+cs}$. Then

$$\sum_{i \in P_k} AC_i\left(N, v, P\right) = \varphi_k\left(M, v^P\right) = AC_k\left(M, v^P, P^m\right).$$

(b) (Uniqueness) Let us assume for a while that two coalitional Shapley values g^1 and g^2 satisfy proportionality within unions (A10) and the quotient game property (A8). Then we can find a game (N, v) and a coalition structure P on N such that $g^1(N, v, P) \neq g^2(N, v, P)$, i.e., $g_i^1(N, v, P) \neq g_i^2(N, v, P)$ for some $i \in N$.

Let us take $P_k \in P$ such that $i \in P_k$. Two possible cases arise.

- $|P_k| = 1$. Then, $P_k = \{i\}$. By A8 we have

$$g_i^1(N, v, P) = g_k^1(M, v^P, P^m) \quad \text{and} \quad g_i^2(N, v, P) = g_k^2(M, v^P, P^m).$$

  Since g^1 and g^2 are coalitional Shapley values

$$g_k^1(M, v^P, P^m) = \varphi_k(M, v^P) = g_k^2(M, v^P, P^m).$$

  Therefore $g_i^1(N, v, P) = g_i^2(N, v, P)$, a contradiction.

- $|P_k| > 1$. Then, there exist two distinct players $i, j \in P_k$. We can assume that j is a non null player. Otherwise, if all players in P_k were null, by the quotient game property and since both solutions are coalitional Shapley values

$$\sum_{i \in P_k} g_i^1\left(N, v, P\right) = \sum_{i \in P_k} g_i^2\left(N, v, P\right) = \varphi_k(M, v^P) = 0$$

  and then $g_i^1\left(N, v, P\right) = g_i^2\left(N, v, P\right) = 0$ for all $i \in P_k$.

  Then, let j be a non null player. As both solutions coincide with the Shapley value when the system of unions is the trivial one, we have $g_i^l\left(N, v, P^n\right) = \varphi_i\left(N, v\right)$ and $g_j^l\left(N, v, P^n\right) = \varphi_j\left(N, v\right) > 0$ for all $l \in \{1, 2\}$. By A10,

$$g_i^l\left(N, v, P\right) g_j^l\left(N, v, P^n\right) = g_i^l\left(N, v, P^n\right) g_j^l\left(N, v, P\right) \quad \text{for all} \quad l \in \{1, 2\}.$$

Suppose that $g_j^1(N,v,P)=0$. Then $g_i^2(N,v,P)=0$ for all $i\in P_k$ and hence $\sum_{i\in P_k} g_i^1(N,v,P)=\varphi_k(M,v^P)=0=\sum_{i\in P_k} g_i^2(N,v,P)$, so that $g_i^2(N,v,P)=0$ for all $i\in P_k$. A similar argument is valid if we suppose that $g_j^2(N,v,P)=0$.

Finally, suppose that $g_j^l(N,v,P)>0$ for all $l\in\{1,2\}$. Two possible cases arise.

i) $g_i^1(N,v,P)=0$ implies $g_i^2(N,v,P)=0$ and, conversely, $g_i^2(N,v,P)=0$ implies $g_i^1(N,v,P)=0$.

ii) $g_i^l(N,v,P)>0$ for any $l\in\{1,2\}$. Then it holds that

$$\frac{g_i^1(N,v,P)}{g_i^2(N,v,P)}=c_k \quad \text{for all} \quad i\in P_k'$$

(where c_k is a constant and P_k' is the subset of P_k formed by the non null players of P_k) and therefore

$$\sum_{i\in P_k} g_i^1(N,v,P)-\sum_{i\in P_k} g_i^2(N,v,P)=\sum_{i\in P_k} c_k g_i^2(N,v,P)-\sum_{i\in P_k} g_i^2(N,v,P)=0.$$

But this implies that $c_k=1$, that is, $g_i^1(N,v,P)=g_i^2(N,v,P)$ for all $i\in P_k$. $\square$.

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